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# LIMIT CYCLES IN A SWITCHING LIÉNARD SYSTEM

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Abstract. In this paper, we consider a class of quadratic switching Liénard systems with three switching lines. We give an algorithm for computing the Lyapunov constants of this system. Based on this method, we obtain a center condition and three limit cycles bifurcating from the focus $(0, 0)$. Further, an example of quadratic switching systems is constructed to show the existence of six limit cycles bifurcating from the center. This is a new low bound on the maximal number of small-amplitude limit cycles obtained in such quadratic switching systems.

1. **Introduction.** In mechanics, engineering and control fields, many mathematical models are described by dynamical systems whose right-hand side are not continuous or not differentiable, see for instance the classical books [1, 12]. Based on the important application of switching systems, the center and limit cycle of switching systems have attracted more and more attention.

Filippov [7] established the qualitative theory for switching systems. Coll and Gasull [4] obtained formulas for computing the first three Lyapunov quantities. Five limit cycles in a quadratic switching system are constructed by Gasull and Torregrosa [8], while only 4 limit cycles exist for quadratic continuous systems [14, 2]. Center conditions have been established for the switching Kukles and Liénard systems [8, 5]. Han and Zhang [11] proved that 2 limit cycles can bifurcate from a focus for piecewise linear systems. Chen and Du [3] constructed a switching Bautin system to show the existence of 9 limit cycles. Tian and Yu [15] provided a complete classification on center conditions in the switching Bautin system, and constructed an example to prove that 10 limit cycles can bifurcate from the center. Recently, a planar quadratic switching system (the switching line isn't straight) has been constructed to obtain 16 limit cycles [6] by using the averaging approach up to $\varepsilon^2$ order. Meanwhile, some works focus on the center and limit cycle problems in cubic

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switching systems. Guo et al. [10] studied a class of $Z_2$-equivariant cubic switching systems, and showed the existence of 18 limit cycles. Very recently, Gouveia and Torregrosa [9] found 24 limit cycles in a cubic switching polynomial system by perturbing a single Darboux center. Yu et al. [18] constructed a cubic switching polynomial system with $Z_2$-symmetry, and proved that such a system could exhibit total 18 limit cycles around symmetric foci.

However, few works focus on the system with more than two switching lines. For example, Wang et al. [16] investigated the limit cycle bifurcations for a class of perturbed planar piecewise smooth systems with 4 switching lines. Li and Yu [13] constructed a cubic $Z_2$-equivariant system with 4 switching lines and proved the existence of 15 limit cycles.

If non-smooth systems have different definitions for the continuous vector fields in two or more different regions divided by lines or curves, we call such systems switching systems. For example, a class of switching system with three lines ($y > 0, x = 0$; $x < 0, y = 0$; $y < 0, x = 0$) is described by

$$(\dot{x}, \dot{y}) = \begin{cases} (\delta x - y + f_1(x,y),\ x + \delta y + g_1(x,y)), & \text{if } x < 0, y < 0, \\ (\delta x - y + f_2(x,y),\ x + \delta y + g_2(x,y)), & \text{if } x > 0, \\ (\delta x - y + f_3(x,y),\ x + \delta y + g_3(x,y)), & \text{if } x < 0, y > 0, \end{cases} \tag{1}$$

where $|\delta| \ll 1$, $f_i(x,y)$ and $g_i(x,y)$, $i = 1,2,3$ are analytic functions in $x$ and $y$, starting from at least second-order terms. Actually, the origin of system (1) is an equilibrium. System (1) includes three subsystems called $S_1$, $S_2$ and $S_3$, which defined for $[x < 0, y < 0]$, $[x > 0]$ and $[x < 0, y > 0]$ respectively.

Many researchers [17] have consider the smooth generalized Liénard system $\ddot{y} + f(y)\dot{y} + g(y) = 0$, which is rewritten as a differential system:

$$\dot{x} = -xf(y) - g(y), \qquad \dot{y} = x.$$

In this paper, we will study a switching Liénard system:

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{cases} \begin{pmatrix} x(\delta + a_{11}y) - y + a_{02}y^2 \\ x \end{pmatrix}, & \text{if} \quad x < 0, y < 0, \\ \begin{pmatrix} -y \\ x \end{pmatrix}, & \text{if} \quad x > 0, \\ \begin{pmatrix} x(\delta + b_{11}y) - y + b_{02}y^2 \\ x \end{pmatrix}, & \text{if} \quad x < 0, y > 0, \end{cases} \tag{2}$$

where $\delta$, $a_i$'s and $b_i$'s are real parameters, satisfying $|\delta| \ll 1$.

To study the center conditions and bifurcation of limit cycles associated with a singular point in a switching system, we need to compute Lyapunov constants of the switching Liénard system (2).

The main goal of this paper is to give an algorithm for computing Lyapunov constants of switching systems with three switching lines. Based on the method, we derive center condition and analyze the bifurcation of limit cycles in (2). We compute the first four Lyapunov constants for the singular point $(0,0)$ of system (2) to obtain the center conditions and prove the existence of 3 limit cycles bifurcating from the origin. Then we choose the center condition with proper perturbation to construct a perturbed system, and then compute the Lyapunov constants associated with the origin to prove the existence of six limit cycles around $(0,0)$.

2. **Computation of Lyapunov constants.** In this section, we give an algorithm for computing Lyapunov constants. Note that by computation we know $\delta$ is the linear perturbation parameter, which will be setted as 0 when we consider limit cycle and center problems. It is observed that system (2) is a special case of (1) by letting $\delta = 0$. Thus we give the general method for computing Lypunov constants of system (1).

We first rewrite the system (1) as

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{cases} \begin{pmatrix} \delta x - y + \sum\limits_{k=2}^{n} X_{1,k}(x,y) \\ x + \delta y + \sum\limits_{k=2}^{n} Y_{1,k}(x,y) \end{pmatrix}, & \text{if } \; x < 0, y < 0, \\ \begin{pmatrix} \delta x - y + \sum\limits_{k=2}^{n} X_{2,k}(x,y) \\ x + \delta y + \sum\limits_{k=2}^{n} Y_{2,k}(x,y) \end{pmatrix}, & \text{if } \; x > 0, \\ \begin{pmatrix} \delta x - y + \sum\limits_{k=2}^{n} X_{3,k}(x,y) \\ x + \delta y + \sum\limits_{k=2}^{n} Y_{3,k}(x,y) \end{pmatrix}, & \text{if } \; x < 0, y > 0, \end{cases} \tag{3}$$

where $X_{i,k}(x,y)$ and $Y_{i,k}(x,y)$ are homogeneous polynomials in $x$ and $y$. Under the polar coordinates transformation, $x$=$r$cos$\theta$ and $y$=$r$sin$\theta$, (3) can be rewritten as

$$\frac{dr}{d\theta} = \begin{cases} \dfrac{\delta r + \sum_{k=2}^{n} \Upsilon_{1,k}(\theta) r^k}{1 + \sum_{k=2}^{n} \Theta_{1,k}(\theta) r^{k-1}} = F_1(r,\theta), & \text{for } \theta \in (-\pi, -\frac{\pi}{2}), \\ \dfrac{\delta r + \sum_{k=2}^{n} \Upsilon_{2,k}(\theta) r^k}{1 + \sum_{k=2}^{n} \Theta_{2,k}(\theta) r^{k-1}} = F_2(r,\theta), & \text{for } \theta \in (-\frac{\pi}{2}, \frac{\pi}{2}), \\ \dfrac{\delta r + \sum_{k=2}^{n} \Upsilon_{3,k}(\theta) r^k}{1 + \sum_{k=2}^{n} \Theta_{3,k}(\theta) r^{k-1}} = F_3(r,\theta), & \text{for } \theta \in (\frac{\pi}{2}, \pi), \end{cases} \tag{4}$$

where $\Upsilon_{i,k}(\theta)$ and $\Theta_{i,k}(\theta)$ are polynomials in sin$\theta$ and cos$\theta$ of degrees $k+1$. By the method of small parameters of Poincaré, the solutions of the three subsystems of (4) are given by

$$r_1(h,\theta) = \sum_{k\geq 1} u_k(\theta) h^k, \qquad r_2(h,\theta) = \sum_{k\geq 1} v_k(\theta) h^k, \qquad r_3(h,\theta) = \sum_{k\geq 1} w_k(\theta) h^k, \tag{5}$$

where

$$\begin{aligned} &u_1(-\tfrac{\pi}{2}) = v_1(-\tfrac{\pi}{2}) = 1, \; w_1(\tfrac{\pi}{2}) = v_1(\tfrac{\pi}{2}), \\ &u_k(-\tfrac{\pi}{2}) = v_k(-\tfrac{\pi}{2}) = 0, \; w_k(\tfrac{\pi}{2}) = v_k(\tfrac{\pi}{2}), \; \forall k \geq 2. \end{aligned} \tag{6}$$

Then, we know that $r_2(h, -\frac{\pi}{2}) = r_1(h, -\frac{\pi}{2}) = h$ and $r_3(h, \frac{\pi}{2}) = r_2(h, \frac{\pi}{2})$.

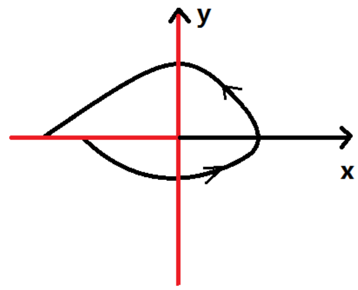


FIGURE 1. Diagram

Substituting the solutions (5) into (4), we can solve $u_k(\theta)$ or $v_k(\theta)$ one by one by integral operations. Consequently, we can define the following successive functions,

$$\triangle_1(h) = h - r_1(h, -\pi),\ \triangle_2(h) = r_2(h, \frac{\pi}{2}) - r_2(h, -\frac{\pi}{2}),\ \triangle_3(h) = r_3(h, \pi) - r_3(h, \frac{\pi}{2})$$

for the three subsystems of (4), respectively. From the diagram (Figure 1), the successive function for the switching system (3) can be defined as

$$\begin{aligned} \triangle(h) &= \triangle_1(h) + \triangle_2(h) + \triangle_3(h) \\ &= h - r_1(h, -\pi) + r_2(h, \tfrac{\pi}{2}) - r_2(h, -\tfrac{\pi}{2}) + r_3(h, \pi) - r_3(h, \tfrac{\pi}{2}) \\ &= r_3(h, \pi) - r_1(h, -\pi). \end{aligned} \tag{7}$$

Thus the displacement function $\triangle(h)$ can be expanded as

$$\triangle(h) = \sum_{k=1}^{n} (w_k(\pi) - u_k(-\pi))h^k = \sum_{k=0}^{n-1} V_k h^{k+1}, \tag{8}$$

where $V_k$ is called the $k$th-order Lyapunov constant of the switching system (3).

Based on the above theory, we give an algorithm for computing the Lyapunov constants.

Next, we turn to discuss how to determine the maximal number of limit cycles which may bifurcate from a Hopf critical point. Generally, the following theorem gives sufficient conditions for the existence of small-amplitude limit cycles in the switching system (1). (The proof can be found in [15].)

**Theorem 2.1.** *Suppose that there exists a sequence of Lyapunov constants of system* (3), $V_{i_0}$, $V_{i_1}$, $\ldots$, $V_{i_k}$, *with* $1 = i_0 < i_1 < \cdots < i_k$, *such that* $V_j = O(|V_{i_0}, \ldots, V_{i_l}|)$ *for any* $i_l < j < i_{l+1}$. *Further, if at the critical point* $\boldsymbol{C}$, $V_{i_0} = V_{i_1} = \cdots = V_{i_k-1} = 0$, $V_{i_k} \neq 0$, *and*

$$\det \left[ \frac{\partial(V_{i_0}, V_{i_1}, \cdots, V_{i_{k-1}})}{\partial(c_1, c_2, \cdots, c_k)} \right]_{\boldsymbol{C}} \neq 0, \tag{9}$$

*then system* (3) *has exactly* $k$ *limit cycles in a* $\delta$*-ball with its center at the origin.*

3. **Center condition and Hopf bifurcation for system (2).** In this section, we consider the center conditions and bifurcation of limit cycles for the switching quadratic Liénard system (2). Clearly, the singular point $(0, 0)$ of system (2) is a Hopf-type critical point. In the following, we first use Algorithm 1 to compute the Lyapunov constants for the origin of system (2), and then use them to derive the center conditions and consider limit cycle bifurcation.

**Algorithm 1** Lyapunov Constants Algorithm

**Input:** System (1).
**Output:** Lyapunov constants.

Rewrite system (1) as (4).
Let $\widehat{F}_i(h,\theta)$ be the expression by substituting (5) to $F_i(r,\theta)$ for $i=1,2,3$ in (4).
Expand $\widehat{F}_i(h,\theta)$ in Taylor series at $h=0$, denoted by $\widehat{F}_i(h,\theta)=\sum_{j=1}^{\infty} P_{i,j}(\theta)h^j$.
Let $E_{1,k}$ be the ordinary equation $\frac{du_k(\theta)}{d\theta}=P_{1,k}(\theta)$, $E_{2,k}$ be the ordinary equation $\frac{dv_k(\theta)}{d\theta}=P_{2,k}(\theta)$, $E_{3,k}$ be the ordinary equation $\frac{dw_k(\theta)}{d\theta}=P_{3,k}(\theta)$.
**for** k from 1 to n+1 **do**
    Solve $u_k(\theta)$ from $E_{1,k}$, $v_k(\theta)$ from $E_{2,k}$ and $w_k(\theta)$ from $E_{3,k}$ under the initial conditions (6).
**end for**
**for** k from 0 to n **do**
    Let $V_k=w_{k+1}(\pi)-u_{k+1}(-\pi)$.
**end for**
**return** $V_k$.

With the aid of the program in Maple, we have computed the Lyapunov constants associated with the singular points $(0,0)$ of system (2), as given in the following theorem.

**Theorem 3.1.** *For system (2), the first four Lyapunov constants at the origin are given by*

$$\begin{aligned} V_0 &= \pi\delta + O(\delta^2),\\ V_1 &= \tfrac{1}{3}(b_{11}-b_{02}-a_{02}-a_{11}),\\ V_2 &= \tfrac{1}{18}(a_{02}^2-b_{02}^2)-\tfrac{1}{36}(a_{11}^2-b_{11}^2)+(\tfrac{\pi}{16}-\tfrac{2}{9})(a_{02}a_{11}+b_{02}b_{11}),\\ V_3 &= \tfrac{1}{135}(a_{11}^3-b_{11}^3)+(\tfrac{\pi}{8}-\tfrac{16}{45})(a_{02}^2a_{11}-b_{02}^2b_{11}). \end{aligned}$$

Now, we turn to discuss the Hopf bifurcation of system (2). From Theorem 3.1 we have the following two theorems.

**Theorem 3.2.** *The origin of system (2) is a center if and only if $\delta=0$, $b_{11}=a_{11}$ and $b_{02}=-a_{02}$.*

**Theorem 3.3.** *System (2) can have at least 3 limit cycles around the origin.*

3.1. **Proof of Theorems 3.2 and 3.3.** For system (2), as discussed in the proof of Theorem 3.1, we set $\delta=0$ to get $V_0=0$. From the second Lyapunov constant $V_1$ in Theorem 3.1, we solve $V_1=0$ to obtain

$$b_{02}=b_{11}-a_{02}-a_{11}. \tag{10}$$

Then, solving $V_2=0$ yields

$$a_{02}=\frac{3\pi b_{11}+4a_{11}-12b_{11}}{3\pi-16}. \tag{11}$$

Then we have

$$V_3=-\frac{a_{11}-b_{11}}{1080(3\pi-16)^2}V_{31}, \tag{12}$$

where

$$\begin{aligned} V_{31} &= (4096 - 1392\pi - 72\pi^2)(a_{11}^2 + b_{11}^2) - (88064 - 67872\pi + 16488\pi^2 \\ &\quad -1215\pi^3)a_{11}b_{11}. \end{aligned} \tag{13}$$

Then, we solve $V_3 = 0$. We give the following discussion.

1) Let $a_{11} - b_{11} = 0$ to solve $V_3 = 0$.

Then we have $b_{11} = a_{11}$ and $b_{02} = -a_{02}$ by (10)(11). Under this condition, system (2) becomes

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{cases} \begin{pmatrix} -y + a_{11}xy + a_{02}y^2 \\ x \end{pmatrix}, & \text{if } x < 0, y < 0, \\ \begin{pmatrix} -y \\ x \end{pmatrix}, & \text{if } x > 0,, \\ \begin{pmatrix} -y + a_{11}xy - a_{02}y^2 \\ x \end{pmatrix}, & \text{if } x < 0, y > 0. \end{cases} \tag{14}$$

It is immediately observed that system (14) remains unchanged under the transformation $[x \to x,\ y \to -y,\ t \to -t]$, which implies (14) is symmetric with the $x$-axis. Thus, the origin is a center.

2) It is shown that $V_{31}$ has the discriminant,

$$\Delta = -3(16 - 3\pi)^2(135\pi - 376)(26624 - 21696\pi + 5544\pi^2 - 405\pi^3) < 0,$$

which implies that $V_{31} = 0$ has no solutions besides $a_{11} = b_{11} = 0$. However, $a_{11} = b_{11} = 0$ results $b_{02} = a_{02} = 0$, which makes system (2) become the trivial integral system $\dot{x} = -y,\ \dot{y} = x$. Thus the origin is a center, and this is a special case of 1).

From the case 1), Theorem 3.2 is proved.

Next, for obtaining the maximum number of limit cycles, we set $a_{11} \neq b_{11}$. With the results obtained above, a direct calculation shows that the determinant evaluated at the critical values is given by

$$\det\left[\frac{\partial(V_1, V_2)}{\partial(b_{02}, a_{02})}\right] = -\frac{(a_{11} - b_{11})(3\pi - 16)}{144} \neq 0.$$

From case 2), by Theorem 2.1, system (2) have 3 small-amplitude limit cycles bifurcating from the center-type singular point (the origin). Theorem 3.3 is proved.

4. **Six limit cycles generated by perturbing system (2) under the center condition.** In this section, we present our main result of this paper. We want to perturb the system (2) to generate small-amplitude limit cycles around the center. We add quadratic perturbations to system (2) to obtain the following perturbed system:

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{cases} \begin{bmatrix} -y + a_{11}xy + a_{02}y^2 + \epsilon(\delta x + p_1x^2 + p_2xy + p_3y^2) \\ x + \epsilon(\delta y + p_4x^2 + p_5xy + p_6y^2) \end{bmatrix}, & \text{if } x < 0, y < 0, \\ \begin{bmatrix} -y \\ x \end{bmatrix}, & \text{if } x > 0, \\ \begin{bmatrix} -y + a_{11}xy - a_{02}y^2 + \epsilon(\delta x + q_1x^2 + q_2xy + q_3y^2) \\ x + \epsilon(\delta y + q_4x^2 + q_5xy + q_6y^2) \end{bmatrix}, & \text{if } x < 0, y > 0, \end{cases} \tag{15}$$

where $\delta$, $p_i$'s and $q_i$'s are real parameters, satisfying $|\delta| \ll 1$ and $0 < \epsilon \ll 1$.

**Theorem 4.1.** *The perturbed system* (15) *can have at least six limit cycles around the origin.*

*Proof.* To prove the existence of six limit cycles, we need to find the $\epsilon$-order Lyapunov constants $\epsilon\mathcal{V}_i,\ i=0,1,\cdots$. First we have $\mathcal{V}_0=\pi\delta$, thus letting $\delta=0$ yields $\mathcal{V}_0=0$. Then we obtain

$$\mathcal{V}_1=\frac{1}{3}(q_2+q_4+2q_6-2q_1-q_3-q_5-2p_1-p_3-p_5-p_2-p_4-2p_6). \tag{16}$$

Solving $\mathcal{V}_1$ for $p_6$ to get

$$p_6=q_6-q_1-p_1+\frac{1}{2}(q_2+q_4-q_3-q_5-p_3-p_5-p_2-p_4). \tag{17}$$

Now, in order to solve higher order Lyapunov constant equations using the remaining perturbation parameters, we assume that $F_0=a_{11}F_1F_2\neq 0$, where

$$\begin{aligned} F_1\ &= 45\pi a_{02}^2-6\pi a_{02}a_{11}-128a_{02}^2+48a_{11}^2,\\ F_2\ &= 103275\pi^3a_{02}^3a_{11}^2-275400\pi^2a_{02}^4a_{11}-752760\pi^2a_{02}^3a_{11}^2+1309500\pi^2a_{02}^2a_{11}^3\\ &\quad -181440\pi^2a_{02}a_{11}^4-115200\pi a_{02}^5+2031360\pi a_{02}^4a_{11}-68544\pi a_{02}^3a_{11}^2-4412160\pi a_{02}^2a_{11}^3\\ &\quad +1133088\pi a_{02}a_{11}^4-112320\pi a_{11}^5+327680a_{02}^5-3735552a_{02}^4a_{11}+4300800a_{02}^3a_{11}^2\\ &\quad +843776a_{02}^2a_{11}^3-1658880a_{02}a_{11}^4+208896a_{11}^5. \end{aligned} \tag{18}$$

Then, solving $\mathcal{V}_2$, $\mathcal{V}_3$ and $\mathcal{V}_4$ for $p_2,\ p_4,\ p_3$ respectively, we obtain

$$\begin{aligned} p_2\ &= \tfrac{1}{6a_{11}}(6\pi a_{02}p_1+3\pi a_{02}p_3+3\pi a_{02}p4+3\pi a_{02}p5+6\pi a_{02}q_1+3\pi a_{02}q_3-3\pi a_{02}q_4\\ &\quad +3\pi a_{02}q_5-3\pi a_{11}p_1-3\pi a_{11}p_3-3\pi a_{11}q_1-3\pi a_{11}q_3-8a_{02}p_1-16a_{02}p_3-4a_{02}p_5\\ &\quad -8a_{02}q_1-16a_{02}q_3-4a_{02}q_5+4a_{11}p_1+2a_{11}p_3+6a_{11}p_4+2a_{11}p_5+4a_{11}q_1+6a_{11}q_2\\ &\quad +2a_{11}q_3-6a_{11}q_4+2a_{11}q_5),\\ p_4\ &= -\tfrac{1}{F_1}(90\pi a_{02}^2p_1+45\pi a_{02}^2p_3+45\pi a_{02}^2p_5+90\pi a_{02}^2q_1+45\pi a_{02}^2q_3-45\pi a_{02}^2q_4\\ &\quad +45\pi a_{02}^2q_5-147\pi a_{02}a_{11}p_1-96\pi a_{02}a_{11}p_3-51\pi a_{02}a_{11}p_5-147\pi a_{02}a_{11}q_1\\ &\quad -96\pi a_{02}a_{11}q_3+6\pi a_{02}a_{11}q_4-51\pi a_{02}a_{11}q_5+6\pi a_{11}^2p_1+6\pi a_{11}^2p_3+6\pi a_{11}^2q_1\\ &\quad +6\pi a_{11}^2q_3-256a_{02}^2p_1-128a_{02}^2p_3-128a_{02}^2p_5-256a_{02}^2q_1-128a_{02}^2q_3+128a_{02}^2q_4\\ &\quad -128a_{02}^2q_5+448a_{02}a_{11}p_1+288a_{02}a_{11}p_3+160a_{02}a_{11}p_5+448a_{02}a_{11}q_1+288a_{02}a_{11}q_3\\ &\quad +160a_{02}a_{11}q_5-16a_{11}^2p_1-8a_{11}^2p_3-8a_{11}^2p_5-16a_{11}^2q_1-8a_{11}^2q_3-48a_{11}^2q_4-8a_{11}^2q_5), \end{aligned} \tag{19}$$

$$\begin{aligned} p_3\ &= -\tfrac{1}{F_2}(206550\pi^3a_{02}^3a_{11}^2p_1+103275\pi^3a_{02}^3a_{11}^2p_5+206550\pi^3a_{02}^3a_{11}^2q_1+103275\pi^3a_{02}^3a_{11}^2q_3\\ &\quad +103275\pi^3a_{02}^3a_{11}^2q_5-550800\pi^2a_{02}^4a_{11}p_1-275400\pi^2a_{02}^4a_{11}p_5-550800\pi^2a_{02}^4a_{11}q_1\\ &\quad -275400\pi^2a_{02}^4a_{11}q_3-275400\pi^2a_{02}^4a_{11}q_5-1505520\pi^2a_{02}^3a_{11}^2p_1-752760\pi^2a_{02}^3a_{11}^2p_5\\ &\quad -1505520\pi^2a_{02}^3a_{11}^2q_1-752760\pi^2a_{02}^3a_{11}^2q_3-752760\pi^2a_{02}^3a_{11}^2q_5+2084400\pi^2a_{02}^2a_{11}^3p_1\\ &\quad +774900\pi^2a_{02}^2a_{11}^3p_5+2084400\pi^2a_{02}^2a_{11}^3q_1+1309500\pi^2a_{02}^2a_{11}^3q_3+774900\pi^2a_{02}^2a_{11}^3q_5\\ &\quad -181440\pi^2a_{02}a_{11}^4p_1-181440\pi^2a_{02}a_{11}^4q_1-181440\pi^2a_{02}a_{11}^4q_3-230400\pi a_{02}^5p_1\\ &\quad -115200\pi a_{02}^5p_5-230400\pi a_{02}^5q_1-115200\pi a_{02}^5q_3-115200\pi a_{02}^5q_5+4062720\pi a_{02}^4a_{11}p_1 \end{aligned}$$

$$
\begin{aligned}
&+2031360\pi a_{02}^4 a_{11} p_5 + 4062720\pi a_{02}^4 a_{11} q_1 + 2031360\pi a_{02}^4 a_{11} q_3 + 2031360\pi a_{02}^4 a_{11} q_5 \\
&-137088\pi a_{02}^3 a_{11}^2 p_1 - 68544\pi a_{02}^3 a_{11}^2 p_5 - 137088\pi a_{02}^3 a_{11}^2 q_1 - 68544\pi a_{02}^3 a_{11}^2 q_3 \\
&-68544\pi a_{02}^3 a_{11}^2 q_5 - 7182720\pi a_{02}^2 a_{11}^3 p_1 - 2770560\pi a_{02}^2 a_{11}^3 p_5 - 7182720\pi a_{02}^2 a_{11}^3 q_1 \\
&-4412160\pi a_{02}^2 a_{11}^3 q_3 - 2770560\pi a_{02}^2 a_{11}^3 q_5 + 1851456\pi a_{02} a_{11}^4 p_1 + 718368\pi a_{02} a_{11}^4 p_5 \\
&+1851456\pi a_{02} a_{11}^4 q_1 + 1133088\pi a_{02} a_{11}^4 q_3 + 718368\pi a_{02} a_{11}^4 q_5 - 112320\pi a_{11}^5 p_1 \\
&-112320\pi a_{11}^5 q_1 - 112320\pi a_{11}^5 q_3 + 655360 a_{02}^5 p_1 + 327680 a_{02}^5 p_5 + 655360 a_{02}^5 q_1 \\
&+327680 a_{02}^5 q_3 + 327680 a_{02}^5 q_5 - 7471104 a_{02}^4 a_{11} p_1 - 3735552 a_{02}^4 a_{11} p_5 - 7471104 a_{02}^4 a_{11} q_1 \\
&-3735552 a_{02}^4 a_{11} q_3 - 3735552 a_{02}^4 a_{11} q_5 + 8601600 a_{02}^3 a_{11}^2 p_1 + 4300800 a_{02}^3 a_{11}^2 p_5 \\
&+8601600 a_{02}^3 a_{11}^2 q_1 + 4300800 a_{02}^3 a_{11}^2 q_3 + 4300800 a_{02}^3 a_{11}^2 q_5 + 1687552 a_{02}^2 a_{11}^3 p_1 \\
&+843776 a_{02}^2 a_{11}^3 p_5 + 1687552 a_{02}^2 a_{11}^3 q_1 + 843776 a_{02}^2 a_{11}^3 q_3 + 843776 a_{02}^2 a_{11}^3 q_5 \\
&-3317760 a_{02} a_{11}^4 p_1 - 1658880 a_{02} a_{11}^4 p_5 - 3317760 a_{02} a_{11}^4 q_1 - 1658880 a_{02} a_{11}^4 q_3 \\
&-1658880 a_{02} a_{11}^4 q_5 + 417792 a_{11}^5 p_1 + 208896 a_{11}^5 p_5 + 417792 a_{11}^5 q_1 + 208896 a_{11}^5 q_3 \\
&+208896 a_{11}^5 q_5).
\end{aligned}
\tag{20}
$$

Now, for the above solutions, higher Lyapunov constants are obtained as follows:

$$
\mathcal{V}_5 = \frac{a_{11}^3(p_1+p_5+q_1+q_5)}{75600F_2}\mathcal{V}_{51}, \quad \mathcal{V}_6 = \frac{a_{11}^3(p_1+p_5+q_1+q_5)}{7257600F_2}, \tag{21}
$$

where

$$
\begin{aligned}
\mathcal{V}_{51} &= 1368241875\pi^4 a_{02}^4 a_{11}^2 - 873888750\pi^4 a_{02}^3 a_{11}^3 + 1637212500\pi^3 a_{02}^5 a_{11} \\
&-7388366400\pi^3 a_{02}^4 a_{11}^2 + 4650108750\pi^3 a_{02}^3 a_{11}^3 - 602883000\pi^3 a_{02}^2 a_{11}^4 + 1746360000\pi^2 a_{02}^6 \\
&-20053137600\pi^2 a_{02}^5 a_{11} + 15308559360\pi^2 a_{02}^4 a_{11}^2 - 7723408320\pi^2 a_{02}^3 a_{11}^3 \\
&+2924456400\pi^2 a_{02}^2 a_{11}^4 - 104900400\pi^2 a_{02} a_{11}^5 - 9722956800\pi a_{02}^6 \\
&+93514199040\pi a_{02}^5 a_{11} - 72499908096\pi a_{02}^4 a_{11}^2 + 9961943040\pi a_{02}^3 a_{11}^3 \\
&+309710592\pi a_{02}^2 a_{11}^4 - 109900800\pi a_{02} a_{11}^5 + 25415040\pi a_{11}^6 + 12918456320 a_{02}^6 \\
&-147270402048 a_{02}^5 a_{11} + 173518356480 a_{02}^4 a_{11}^2 - 11920211968 a_{02}^3 a_{11}^3 - 11890851840 a_{02}^2 a_{11}^4 \\
&+1497366528 a_{02} a_{11}^5, \\
\mathcal{V}_{61} &= 11366932500\pi^5 a_{02}^4 a_{11}^3 + 230285632500\pi^4 a_{02}^5 a_{11}^2 + 18003384000\pi^4 a_{02}^4 a_{11}^3 \\
&-92960713125\pi^4 a_{02}^3 a_{11}^4 + 817249702500\pi^3 a_{02}^6 a_{11} - 1643497516800\pi^3 a_{02}^5 a_{11}^2 \\
&-107742135825\pi^3 a_{02}^4 a_{11}^3 + 388552140000\pi^3 a_{02}^3 a_{11}^4 - 57514388175\pi^3 a_{02}^2 a_{11}^5 \\
&+401163840000\pi^2 a_{02}^7 - 7740994435200\pi^2 a_{02}^6 a_{11} + 5323022745120\pi^2 a_{02}^5 a_{11}^2 \\
&-629941730400\pi^2 a_{02}^4 a_{11}^3 - 481856099400\pi^2 a_{02}^3 a_{11}^4 + 280778162400\pi^2 a_{02}^2 a_{11}^5 \\
&-6007332600\pi^2 a_{02} a_{11}^6 - 2037216153600\pi a_{02}^7 + 24997831495680\pi a_{02}^6 a_{11} \\
&-15319006313472\pi a_{02}^5 a_{11}^2 - 1487806424064\pi a_{02}^4 a_{11}^3 + 1648446825984\pi a_{02}^3 a_{11}^4 \\
&-200769679872\pi a_{02}^2 a_{11}^5 - 9968094720\pi a_{02} a_{11}^6 + 2504040960\pi a_{11}^7 + 2480343613440 a_{02}^7 \\
&-27449135988736 a_{02}^6 a_{11} + 23890218713088 a_{02}^5 a_{11}^2 + 8816494116864 a_{02}^4 a_{11}^3 \\
&-3045937119232 a_{02}^3 a_{11}^4 - 473520144384 a_{02}^2 a_{11}^5 + 95831457792 a_{02} a_{11}^6.
\end{aligned}
\tag{22}
$$

To obtain the maximal number of limit cycles in system (15), we assume $p_1 + p_5 + q_1 + q_5 \neq 0$. We first solve $\mathcal{V}_{51}$ to obtain the solutions $a_{02} = z^* a_{11}$, where $z^*$ is a solution of

$$\begin{aligned}
&(1746360000\pi^2 - 9722956800\pi + 12918456320)z^6 + 25415040\pi - (104900400\pi^2 + 109900800\pi \\
&-1497366528)z + (2924456400\pi^2 - 602883000\pi^3 + 309710592\pi - 11890851840)z^2 \\
&+(4650108750\pi^3 - 873888750\pi^4 - 7723408320\pi^2 + 9961943040\pi - 11920211968)z^3 \\
&+(1368241875\pi^4 - 7388366400\pi^3 + 15308559360\pi^2 - 72499908096\pi + 173518356480)z^4 \\
&+(1637212500\pi^3 - 20053137600\pi^2 + 93514199040\pi - 147270402048)z^5 = 0.
\end{aligned} \tag{23}$$

It is easily verified that the equation (23) has two solutions: $-0.2160073381\cdots$, $1.656188526\cdots$. Then we compute

$$\det\left[\frac{\partial(\mathcal{V}_1, \mathcal{V}_2, \mathcal{V}_3, \mathcal{V}_4, \mathcal{V}_5)}{\partial(p_6, p_2, p_4, p_3, a_{02})}\right] = \frac{(p_1 + p_5 + q_1 + q_5)a_{11}^4 F_{\det}}{33861058560000 F_2},$$

and

$$\begin{aligned}
\text{resultant}(\mathcal{V}_{51}, \mathcal{V}_{61}, a_{02}) &= c_1 \times 10^{139} a_{11}^{42}, \\
\text{resultant}(\mathcal{V}_{51}, F_{det}, a_{02}) &= c_2 \times 10^{204} a_{11}^{60}, \\
\text{resultant}(\mathcal{V}_{51}, F_0, a_{02}) &= c_3 \times 10^{111} a_{11}^{48},
\end{aligned} \tag{24}$$

where $c_1$, $c_2$ and $c_3$ are non-zero constants. By Theorem 2.1, system (15) have 6 small-amplitude limit cycles around the origin. Theorem 4.1 is proved. □

## 5. Conclusion.

In this paper, we considered a class of planar switching quadratic Liénard systems, and gave an algorithm for computing the Lyapunov constants of the planar switching systems with three switching lines. We obtained the center condition and proved the existence of 3 limit cycles using the algorithm with the aid of Maple. We further constructed a perturbed system, and proved the existence of 6 limit cycles around the origin. The existence of 6 limit cycles is a new lower bound on the maximal number of small-amplitude limit cycles obtained around one singular point in such switching systems with three switching lines.

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